

MAS212 Linear Algebra I 2007-2008

Coursework 7 — Solutions

Exercise* 1. Since

$$\begin{vmatrix} 1 & -3 & 2 \\ 0 & 1 & 0 \\ -4 & 12 & -7 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 2 \\ -4 & -7 \end{vmatrix} = 1 \cdot (-7 - (-8)) = 1 \neq 0,$$

the three vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are linearly independent, by Theorem 4.31. But since $\dim \mathbb{R}^3 = 3$, they form a basis for $\mathbb{R}^3$, by Theorem 4.45.

Similarly, since

$$\begin{vmatrix} 1 & -1 & 0 \\ 4 & 1 & 2 \\ 2 & 3 & 0 \end{vmatrix} = (-2) \cdot \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = (-2) \cdot (3 - (-2)) = -10 \neq 0,$$

the three vectors $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ are linearly independent so must form a basis for $\mathbb{R}^3$.

Exercise* 2.

(a) By Exercise 2 (b) of Coursework 6, the three vectors $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are linearly independent. But since $\dim P_2 = 3$, they form a basis for P_2 , by Theorem 4.45.

(b) We need to find $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$\mathbf{p} = c_1 \mathbf{p}_1 + c_2 \mathbf{p}_2 + c_3 \mathbf{p}_3.$$

This means that, for all t

$$1 - t + 3t^2 = c_1(1 - t^2) + c_2 t + c_3(t + 2t^2),$$

so, for all t

$$1 - t + 3t^2 = c_1 + (c_2 + c_3)t + (-c_1 + 2c_3)t^2.$$

Thus, comparing coefficients of the polynomials on the right and on the left of the equation above, the c_i need to satisfy the system

$$\begin{array}{rcl} c_1 & & = 1 \\ & c_2 + c_3 & = -1 \\ -c_1 & + 2c_3 & = 3 \end{array}$$

We could proceed using Gaussian elimination, but the system is easily solved by inspection: equation 1 yields $c_1 = 1$; equation 3 then gives $c_3 = (3 + 1)/2 = 2$, and equation 2 finally gives $c_2 = -1 - 2 = -3$. Thus

$$[\mathbf{p}]_{\mathcal{B}} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}.$$

Exercise* 3. Let $\mathcal{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$ and $\mathcal{D} = [\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3]$ be the ordered bases given in Exercise 1.

(a) The transition matrix from $\mathcal{D}$ to the standard basis is the matrix whose columns are formed by the elements of $\mathcal{D}$. Thus

$$S_1 = P_{\mathcal{D}} = \begin{pmatrix} 1 & -1 & 0 \\ 4 & 1 & 2 \\ 2 & 3 & 0 \end{pmatrix}.$$

- (b) The transition matrix from the standard basis to $\mathcal{B}$ is the inverse of the transition matrix $P_{\mathcal{B}}$ from $\mathcal{B}$ to the standard basis. Using Gauss-Jordan inversion we find

$$\begin{aligned} (P_{\mathcal{B}}|I) &= \left(\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -4 & 12 & -7 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_3 + 4R_1} \left(\begin{array}{ccc|ccc} 1 & -3 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 & 1 \end{array} \right) \\ &\xrightarrow{R_1 - 2R_3} \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & -7 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 & 1 \end{array} \right) \xrightarrow{R_1 + 3R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 3 & -2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 4 & 0 & 1 \end{array} \right), \end{aligned}$$

so

$$S_2 = P_{\mathcal{B}}^{-1} = \begin{pmatrix} -7 & 3 & -2 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}.$$

Finally, the transition matrix from $\mathcal{D}$ to $\mathcal{B}$ is

$$P_{\mathcal{B}}^{-1}P_{\mathcal{D}} = \begin{pmatrix} -7 & 3 & -2 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 4 & 1 & 2 \\ 2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 6 \\ 4 & 1 & 2 \\ 6 & -1 & 0 \end{pmatrix}.$$

(c)

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}P_{\mathcal{D}}[\mathbf{x}]_{\mathcal{D}} = \begin{pmatrix} 1 & 4 & 6 \\ 4 & 1 & 2 \\ 6 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -5 \\ 9 \\ 17 \end{pmatrix}.$$

Exercise 4. Suppose that the system $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b} \in \mathbb{R}^m$. Then, in particular, the system $A\mathbf{x} = \mathbf{0}$ can have only one solution, namely the trivial one. This, however, means that the columns of A are linearly independent.

Conversely, suppose that the columns of A are linearly independent. Then the only solution of $A\mathbf{x} = \mathbf{0}$ is the trivial solution. Now if $\mathbf{x}_1$ and $\mathbf{x}_2$ are both solutions of $A\mathbf{x} = \mathbf{b}$, then $\mathbf{x}_1 - \mathbf{x}_2$ is a solution of $A\mathbf{x} = \mathbf{0}$ because

$$A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Thus, $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}$, so $\mathbf{x}_1 = \mathbf{x}_2$. In other words, if $A\mathbf{x} = \mathbf{b}$ has a solution, it must be unique.

Exercise* 5. First we bring A to row echelon form:

$$A = \begin{pmatrix} 1 & 2 & -2 & 1 & 3 \\ 1 & 2 & -1 & 0 & 4 \\ 2 & 4 & -3 & 1 & 7 \end{pmatrix} \xrightarrow{\substack{R_2 - R_1 \\ R_3 - R_1}} \begin{pmatrix} 1 & 2 & -2 & 1 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & 2 & -2 & 1 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = U.$$

The leading variables are x_1 and x_3 ; the free variables are x_2, x_4 , and x_5 . Now, the nonzero rows of U form a basis for $\text{row}(A)$, so

$$\left\{ \begin{pmatrix} 1 & 2 & -2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & -1 & 1 \end{pmatrix} \right\}$$

is a basis for $\text{row}(A)$. Furthermore, the columns of A (not of U !) containing the leading variables form a basis for $\text{col}(A)$, so

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ -3 \end{pmatrix} \right\}$$

is a basis for $\text{col}(A)$. In order to determine a basis for $N(A)$ we solve $U\mathbf{x} = \mathbf{0}$ using back substitution: setting $x_2 = \alpha$, $x_4 = \beta$, and $x_5 = \gamma$, we find $x_3 = x_4 - x_5 = \beta - \gamma$, and

$x_1 = -2x_2 + 2x_3 - x_4 - 3x_5 = -2\alpha + \beta - 5\gamma$. Thus, the general solution of $A\mathbf{x} = \mathbf{0}$ is of the form

$$\begin{pmatrix} -2\alpha + \beta - 5\gamma \\ \alpha \\ \beta - \gamma \\ \beta \\ \gamma \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} -5 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix},$$

so

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\},$$

is a basis for $N(A)$.

Since the bases for $\text{row}(A)$ and $N(A)$ found above have two and three elements, respectively,

$$\text{rank } A = 2, \quad \text{and} \quad \text{nul } A = 3.$$

Thus $\text{rank } A + \text{nul } A = 3 + 2 = 5$, that is, the Rank-Nullity Theorem holds for the matrix A (as it should!).

Exercise* 6. First observe that by the Invertible Matrix Theorem, A is invertible if and only if $N(A) = \{\mathbf{0}\}$, that is, A is invertible if and only if $\text{nul } A = 0$. But, by the Rank-Nullity Theorem, $\text{rank } A = n - \text{nul } A$, so A is invertible if and only if $\text{rank } A = n$.

Exercise* 7. In the following, if $\mathbf{x}$ and $\mathbf{y}$ are two arbitrary vectors in $\mathbb{R}^2$, we denote their entries by x_1, x_2 and y_1, y_2 respectively, so that

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{and} \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

In particular, if α and β are scalars, then

$$\alpha\mathbf{x} + \beta\mathbf{y} = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \end{pmatrix}.$$

(a) Since $L(\alpha\mathbf{x}) = (1, \alpha x_2, \alpha x_1)^T$, and $\alpha L(\mathbf{x}) = (\alpha, \alpha x_2, \alpha x_1)^T$, we see that $L(\alpha\mathbf{x}) \neq \alpha L(\mathbf{x})$ unless $\alpha = 1$, so L is not linear.

(b) L is linear, since

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ 0 \\ \alpha x_2 + \beta y_2 \end{pmatrix} = \alpha \begin{pmatrix} x_1 \\ 0 \\ x_2 \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ 0 \\ y_2 \end{pmatrix} = \alpha L(\mathbf{x}) + \beta L(\mathbf{y}).$$

(c) L is linear, since

$$L(\alpha\mathbf{x} + \beta\mathbf{y}) = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \alpha x_2 + \beta y_2 \\ (\alpha x_1 + \beta y_1) - (\alpha x_2 + \beta y_2) \end{pmatrix} = \alpha \begin{pmatrix} x_1 \\ x_2 \\ x_1 - x_2 \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ y_2 \\ y_1 - y_2 \end{pmatrix} = \alpha L(\mathbf{x}) + \beta L(\mathbf{y}).$$

(d) Since $L(\alpha\mathbf{x}) = (\alpha x_1, \alpha^2 x_2^2, 0)^T$, and $\alpha L(\mathbf{x}) = (\alpha x_1, \alpha x_2^2, 0)^T$, we see that $L(\alpha\mathbf{x}) \neq \alpha L(\mathbf{x})$ unless α is either 0 or 1, so L is not linear.